ON THE PROBLEM OF DAMPING OF A LINEAR SYSTEM UNDER MINIMUM CONTROL INTENSITY

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The paper considers the problem of designing a control $u(t)$ which takes a linear system to an equilibrium ststе under the condition that a given control intensity is a minimum.

1. Consider the control system

$$dx / dt = Ax + Bu \tag{1.1}$$

Here x is an n-vector of the phase coordinates $\{x_i\}$ of the system, u is an r-vector of the control forces $\{u_j\}$, A and B are the $(n \times n)$ and $(n \times r)$ matrices $\{a_{ij}\}$ and $\{b_{ij}\}$, respectively. Let there be given an initial state x^0 of system (1.1), a designated time interval $0 \leqslant t \leqslant T$, a selected class U of functions $u(t)$, and an estimate of control efficiency $\xi[u(\tau)]\,(0 \leqslant \tau \leqslant T)$.

The problem consists of choosing the control $u^0(t)$ which takes system (1.1) from the state $x(0) = x^0$ to the state $x(T) = 0$ and which satisfies condition

$$\xi\,[u^\circ(\tau)] = \min_u \xi\,[u(\tau)] \quad \text{for } u \text{ from } U \tag{1.2}$$

The problem being considered is related to a group of optimum control problems and can be solved by one of the well-known methods in the theory of optimum processes, which have been worked out with sufficient completeness for the linear systems(1.1). Replacing t by $-t$, the conditions of the problem can be transformed so that $x(0) = 0$, $x(T) = x^0$. We shall discuss precisely such a problem.

Let $f_{ij}(t)$ be the elements of the fundamental matrix $F(t)$ of the solutions of the homogeneous system (1.1). The coordinates $x_i(T)$ of the motion of (1.1)

$$x_i\,(T) = \int_0^T h^{(i)}\,(\tau)\cdot u\,(\tau)\,d\tau$$

$$h^{(i)}\,(\tau) = \left\{ h_j^{(i)}\,(\tau) = \sum_{k=1}^n f_{ik}\,(T-\tau)\,b_{kj} \right\} \qquad \binom{i = 1, \ldots, n}{j = 1, \ldots, r} \tag{1.3}$$

are conveniently interpreted as the values of the linear functional

$$\eta_u\,[h\,(\tau)]\ \ (0 \leqslant \tau \leqslant T)\quad x_i\,(T) = \eta_u\,[h^{(i)}\,(\tau)] \qquad (i = 1, \ldots, n) \tag{1.4}$$

generated by the vector-function

$$u\,(\tau) = \{u_j\,(\tau)\} \quad (0 \leqslant \tau \leqslant T,\ j = 1, \ldots, r)$$

Here the symbol $h(\tau)\cdot u(\tau)$ denotes the scalar product of the vectors $\{h_j(\tau)\}$ and $\{u_j(\tau)\}$. Then, the control problem reduces to the problem [1] of constructing the functional η_u° generated by the function $u^\circ(\tau)$ and satisfying conditions (1.2) and (1.4). This problem can be treated as a problem of moments, or as a game, or as a problem of set separation, etc.[2]. Such an approach to the control problem was proposed in paper [3]. The interpretation of the contol problem as a propblem in functional analysis is encountered in various forms in a number of papers. One such approach to the problem is also described below; the optimality criterion which is introduced is not essentially new as compared with the one in [2], however, the form of the criterion presented here has certain useful features.

Let us choose the function $u\,(\tau)$ $(0 \leqslant \tau \leqslant T)$ from those classes U, which generate the linear functionals

$$\eta_u\,[h\,(\tau)] = \int_0^T h\,(\tau)\cdot u\,(\tau)\,d\tau$$

on the vector function $h(\tau)$ for some normed functional space $\{h(\tau)\}$ with a certain norm $\rho[h(\tau)]$. The norm of the functional $\eta_u[h(\tau)]$ will be denoted by the symbol $\rho^*[u]$. The estimate $\xi[u]$ selected for the control problem should be meaningful for functions $u(\tau)$ from U . Further, we shall assume that the following conditions are satisfied.

1) The estimate $\xi[u]$ is positive when $\rho^*[u] > 0$ and the magnitude of $\rho^*[u]$ is uniformly bounded

$$\rho^*\,[u] \leqslant N\,(\beta)\ \text{ when } \xi\,[u] = \beta \quad \text{for all } \beta > 0\ (\xi\,[0] = 0) \tag{1.5}$$

2) For any number $\beta > 0$, if at the elements $h(\tau)$ satisfying the condition

$$\eta_u\,[h\,(\tau)] \leqslant \beta \ \text{ for all } u \text{ from } \xi\,[u] = \beta \tag{1.6}$$

the relation

$$\sup_h\,(\eta_{u^*}\,[h\,(\tau)]) = \beta \tag{1.7}$$

is satisfied, then the inequality

$$\xi\,[u^*] \leqslant \beta \tag{1.8}$$

is valid.

To solve problem (1.2), (1.4) we should consider the set E_β of elements $h(\tau)$ of the form

$$h\,(\tau) = \sum_{i=1}^n l_i h^{(i)}\,(\tau) \tag{1.9}$$

which satisfy condition (1.6). Let us assume that for every β in the interval $0 < \beta < \beta_1$, under conditions (1.6) and (1.9), the quantity $\alpha = l\cdot x^\circ$ has a finite positive maximum

$$\alpha\,(\beta) = \max\, l\cdot x^\circ \tag{1.10}$$

The symbol $h_\beta(\tau)$ denotes the element

$$h_\beta(\tau) = \sum_{i=1}^{n} l_i(\beta)\, h^{(i)}(\tau) \in E_\beta \tag{1.11}$$

at which this maximum is attained. Let the number $\beta^\circ < \beta_1$ satisfy the equality

$$\alpha(\beta^\circ) = \beta^\circ \tag{1.12}$$

and, moreover,

$$\alpha(\beta) > \beta \quad \text{for} \quad 0 < \beta < \beta^\circ \tag{1.13}$$

Then there exists the optimum control $u^\circ(\tau)$ and this control satisfies the condition

$$\eta_{u^\circ}[h^\circ(\tau)] = \max_u(\eta_u[h^\circ(\tau)]) = \beta^\circ \quad \text{for} \quad \xi[u]=\beta^\circ \quad (h^\circ(\tau)=h_{\beta^\circ}(\tau)) \tag{1.14}$$

Indeed, in the space $\{h(\tau)\}$ let us consider the convex sets

$$H = \left\{\sum_{i=1}^{n} l_i h^{(i)}(\tau) \quad \text{when} \quad x^\circ \cdot l = \beta^\circ\right\} \tag{1.15}$$

$$E = \{\eta_u[h(\tau)] \leqslant \beta^\circ \quad \text{for all } u \text{ from } \xi[u]=\beta^\circ\} \tag{1.16}$$

Because of (1.5) the set E contains the ε-neighborhood of the null element $h(\tau) = 0$, where $\varepsilon < \beta^\circ/N(\beta^\circ)$. From the definition of the number $\alpha(\beta)$ in (1.10) and because of equality (1.12), the internal elements $h(\tau)$ from the E in (1.16) are not contained in the H in (1.15). Consequently, the sets H and E satisfy the conditions under which the theorem on the separability of subsets ([1], pp. 443-447) can be used. On the basis of this theorem there exists a linear functional

$$\eta_{u^\circ}[h(\tau)] = \int_0^T h(\tau)\cdot u^\circ(\tau)\, d\tau \tag{1.17}$$

which satisfies conditions

$$\eta_{u^\circ}[h(\tau)] = \beta^\circ \quad \text{for } h(\tau) \text{ from } H \tag{1.18}$$

$$\eta_{u^\circ}[h(\tau)] \leqslant \beta^\circ \quad \text{for } h(\tau) \text{ from } E \tag{1.19}$$

The function $u^\circ(\tau)$ in (1.17) is just an optimum control. In fact it follows from (1.15) and (1.18) that

$$\eta_{u^\circ}[h^{(i)}(\tau)] = x_i^\circ \quad (i = 1, \ldots, n)$$

i.e. condition (1.4) is satisfied. Moreover, from (1.6) to (1.8) and (1.10) to (1.12), (1.15), (1.16), (1.18) and (1.19) it follows that

$$\xi[u^\circ] \leqslant \beta^\circ = \alpha(\beta^\circ) \tag{1.20}$$

There cannot exist a control $u^*(\tau)$ which would solve the control problem for $\xi[u^*] = \beta^* < \beta^\circ$. Indeed, if we assume the contrary, then from (1.4), (1.10) and (1.11) it follows that

$$\eta_{u^*}[h^*(\tau)] = \alpha(\beta^*) \quad (h^* = h_{\beta^*}) \tag{1.21}$$

But $h^*(\tau)$ is contained in E_{β^*} and, consequently, by (1.6) we should have $\eta_{u^*}[h^*(\tau)] \leqslant \xi[u^*] = \beta^*$. This inequality and equality (1.21) contradict (1.13). Now, by the definition of $h^\circ(\tau)$, (1.14) follows from (1.15), (1.16), (1.18), (1.19) and (1.20).

Thus, the control $u^{\circ}(t)$ which has been constructed is really optimum and satisfies condition (1.14).

Note 1.1. An analysis of the reasoning presented above shows that for the given optimality criterion to be valid it suffices for (1.5) to be satisfied only for $\beta = \beta^{\circ}$, since this condition is required only so that the set E in (1.16) may contain the ϵ-neighborhood of the null element $h(\tau) = 0$.

2. The form of the optimality criterion as stated in Section 1 is useful for the following reason. Here we do not require an a priori choice of the basic normed space $\{h(\tau)\}$ so that the quantity $\xi[u]$ defines the norm of the linear functional $\eta_u[h(\tau)]$ on precisely this space, but we need only find the set E_β of elements $h(\tau)$ of form (1.9) satisfying condition (1.6), i.e. condition

$$\int_0^T \Big(\sum_{i=1}^{n} l_i h^{(i)}(\tau)\Big)\cdot u(\tau)\,d\tau \leqslant \xi[u] \quad \text{for} \quad \xi[u] = \beta \tag{2.1}$$

This can sometimes be done from a simpler consideration than the construction of an initial space $\{h(\tau)\}$ with norm $\rho[h]$ which ensures the condition $\rho^*[u] = \xi[u]$. Let us investigate this by means of an example.

Let it be required to take the system

$$dx/dt = Ax + bu \tag{2.2}$$

to equilibrium, where x is a n-vector and u is a scalar, under the condition

$$\xi[u(\tau)] = \max\Big[\max_\tau \varphi(\tau, |u(\tau)|),\ \int_0^T \psi(\tau)|u(\tau)|\,d\tau\Big] = \min \tag{2.3}$$

where $\psi(t)$ and $\varphi(t,y)$ are given functions, positive for $0 \leqslant t \leqslant T$ and for $y > 0$. We shall assume that the functions $\psi(t)$ and $\varphi(t,y)$ are continuous at every t, that the function $\varphi(t,y)$ grows monotonously with y, and that $\lim \varphi(t,y) = \infty$ as $y \to \infty$, $\varphi(t,0) = 0$.

Note 2.1. The assumption of continuity of the functions $\varphi(t,y)$ and $\psi(t)$ is not necessary for carrying out the reasoning described below. The functions $\varphi(t,y)$ and $\psi(t)$ may be discontinuous. It is important only that the function $\omega(t,\beta)$ considered below have the needed measure properties on the interval $[0,T]$.

Thus, we consider the problem of control under the minimality condition and the maximal value of the control force $u(t)$ and of the pulse of this force measured in the scales of $\varphi(t,|u|)$ and $\psi(t)$. As the initial space $\{h(\tau)\}$ let us choose the space of functions $h(\tau)$ which are Lebesgue-integrable on the interval $0 \leqslant \tau \leqslant T$. As the space U of functions $u(\tau)$ let us choose the set of measurable functions $u(\tau)$ almost everywhere bounded on $[0,T]$, since precisely such functions generate the functional $\eta_u[h(\tau)]$ on the functions $h(\tau)$ from the chosen space $\{h(\tau)\}$.

Here [1]

$$\rho[h] = \int_0^T |h(\tau)|\,d\tau \tag{2.4}$$

$$\rho^*[u] = \text{true sup}\,(|u(\tau)| \quad \text{for} \quad 0 \leqslant \tau \leqslant T) \tag{2.5}$$

The quantity $\xi[u]$ in (2.3), for the chosen class U of functions $u(\tau)$ in (2.5), has a meaning only if the quantity $\max_\tau$ on the left-hand side of (2.3) is understood in the sense of a true $\sup_\tau$ ([1] p.115). The estimate $\xi[u]$ satisfies conditions (1) and (2). Indeed, the fulfillment of the conditions $\xi[u] > 0$ when $\rho^*[u] > 0$ and (1.5) is ensured by the properties of the functions $\varphi(t,|u|)$ and $\psi(t)$. We shall check fulfillment of conditions (1.6) to (1.8). Let $u^*(\tau)$ be a function from U satisfying condition (1.8) for the $\xi[u]$ in (2.3) and for $\beta = \beta^*$. This signifies that

$$\text{true}\ \sup_\tau \varphi(\tau, |u^*(\tau)|) = \beta^*, \qquad \int_0^T \psi(\tau)\,|u^*(\tau)|\,d\tau \leqslant \beta^* \tag{2.6}$$

or

$$\int_0^T \psi(\tau)\,|u^*(\tau)|\,d\tau = \beta^*, \qquad \text{true}\ \sup_\tau \varphi(\tau, |u^*(\tau)|) < \beta^* \tag{2.7}$$

Under the asssmptions, for $\beta > 0$ the function $\varphi(t,y) = \beta$ has an inverse continuous function $y = \omega(t,\beta)$, i.e.

$$\varphi(t, \omega(t, \beta)) = \beta \tag{2.8}$$

and for every $t \in [0, T]$ the function $\omega(t,\beta)$ is a monotonously increasing function of β. Let symbol $\mu(t,\beta)$ denote the function

$$\mu(t, \beta) = \frac{1}{\omega(t, \beta)} \tag{2.9}$$

This function is positive and continuous for $\beta > 0,\ 0 \leqslant t \leqslant T$.

Let the function $u^*(t)$ satisfy condition (2.6). For any small $\delta > 0$, under condition (2.6), in the interval $[0,T]$ there is a set Δ_δ with the measure $\mu(\Delta_\delta) > 0$, where $\varphi(\tau, |u^*(\tau)|) > \beta^* - \delta$. On this set the function $|u^*(\tau)| = \omega(\tau, \varphi)$ satisfies condition $\omega(\tau, \varphi) > \omega(\tau, \beta^*) - \varepsilon$, and, moreover, because of the continuity of the considered functions, $\varepsilon \to 0$ as $\delta \to 0$. Let us construct the function $h^\varepsilon(\tau) = \beta\mu(\tau, \beta)\,\mathrm{sign}\,u^* : \mu(\Delta_\delta)$ when τ is from Δ_δ and $h^\varepsilon(\tau) = 0$ when τ is outside Δ_δ. The function $h^\varepsilon(\tau)$ is contained in the set E_β since for any function $u(\tau)$ with true $\sup_\tau \varphi(\tau, |u(\tau)|) \leqslant \beta$, i.e. for any function $u(\tau)$ with true

$$\sup_\tau (|u(\tau)| / \omega(\tau, \beta)) \leqslant 1,$$

the inequality

$$\int_0^T h^\varepsilon(\tau)\, u(\tau)\, d\tau \leqslant \int_{\Delta_\delta} [\beta\mu(\tau, \beta)\,\omega(\tau, \beta) / \mu(\Delta_\delta)]\, d\tau \leqslant \beta \tag{2.10}$$

is valid, and here, if $\beta^* > \beta$, then

$$\int_0^T h^\varepsilon(\tau)\, u^*(\tau)\, d\tau \geqslant \int_{\Delta_\delta} [\beta\mu(\tau, \beta^*)\,[\omega(\tau, \beta^*) - \varepsilon] / \mu(\Delta_\delta)]\, d\tau \geqslant \beta_1 - \varkappa \tag{2.11}$$

Since when $\varepsilon \to 0$ we have $\varkappa \to 0$ and $\beta_1 > \beta$, then from (2.10) and (2.11) we conclude that when $\beta^* > \beta$, (1.7) is not satisfied.

Now let condition (2.7) be satisfied. Any function $h(\tau)$ satisfying condition

$$|h(\tau)| = \psi(\tau)$$

is contained in E_β since then

$$\left|\int_0^T h(\tau)\,u(\tau)\,d\tau\right| \leqslant \int_0^T \psi(\tau)\,|u(\tau)|\,d\tau \leqslant \beta \quad \text{for} \int_0^T \psi(\tau)\,|u(\tau)|\,d\tau \leqslant \beta$$

But when $h(\tau) = \psi(\tau)\operatorname{sign} u^*(\tau)$ and when condition (2.7) is satisfied we have

$$\int_0^T \psi(\tau)\,u^*(\tau)\,[\operatorname{sign} u^*(\tau)]\,d\tau = \int_0^T \psi(\tau)\,|u^*(\tau)|\,d\tau = \beta^* \qquad (2.12)$$

Hence it follows that (1.7) is still not satisfied. Thus the estimate $\xi[u]$ in (2.3) indeed satisfies conditions (1) and (2).

We shall assume that system (2.4) is completely controllable [4]. Then the problem will have a solution.

According to Section 1 we should investigate the set of functions $h(\tau)$ of form (1.8) which satisfy condition (2.1), and for $\beta > 0$ we should find those values of $l_i(\beta)$ for which (1.10) is realized. Condition (2.1) will be satisfied only by such functions $h(\tau)$ in (1.8) which satisfy the condition

$$\int_\Delta \frac{\omega(\tau,\beta)}{\beta}\cdot|h(\tau)|\,d\tau \leqslant 1 \qquad (2.13)$$

for measurable subsets Δ from $[0,T]$ satisfying condition

$$\int_\Delta \frac{\omega(\tau,\beta)}{\beta}\psi(\tau)\,d\tau = 1 \qquad (2.14)$$

in the case when these subsets are contained in the interval $[0,T]$.

However, if the inequality

$$\int_0^T \frac{\omega(\tau,\beta)\,\psi(\tau)}{\beta}\,d\tau \leqslant 1 \qquad (2.15)$$

is fulfilled, then the Δ in (2.13) denotes the interval $[0,T]$.

Hence it follows that the number $\alpha(\beta)$ in (1.10) can be determined from conditions

$$\alpha(\beta) = \frac{1}{\gamma(\beta)} \qquad (2.16)$$

$$\gamma(\beta) = \min_l \max_\Delta \left[\int_\Delta \frac{\omega(\tau,\beta)}{\beta}\left|\sum_{i=1}^n l_i h^{(i)}(\tau)\right| d\tau\right] \quad \text{for } l\cdot x^\circ = 1$$

where the set Δ satisfies condition (2.14) (or coincides with the interval $[0,T]$ if condition (2.15) is fulfilled). If the system is completely controllable, then the quantity $\gamma(\beta)$ depends continuously on β. Because of the properties of the function $\omega(\tau,\beta)$ it follows from (2.16) that for sufficiently small values of β the quantity $\beta\gamma(\beta)$ is arbitrarily small. But this means that for sufficiently small values of β the inequality $\alpha(\beta) > \beta$ is satisfied. Conversely, for sufficiently large values of β the quantity $\beta\gamma(\beta)$ becomes arbitrarily large. Indeed, by assuming the contrary we can obtain sequence $\beta_k \to \infty$, Δ_k, and $\{l_i(\beta_k)\}$ for which

$$\overline{\lim} \int_{\Delta_k} \omega\ (\tau,\ \beta_k) \left| \sum_{i=1}^{n} l_i\ (\beta_k)\ h^{(i)}\ (\tau) \right| d\tau = N < \infty \quad \text{for } k \to \infty \qquad (2.17)$$

would be satisfied.

If the measure $\mu(\Delta_k)$ does not approach zero as $k \to \infty$, then the relation (2.17) is not possible in consequence of $\min_\tau \omega\ (\tau,\ \beta_k) \to \infty$ also by the reason that under the conditions of complete controllability

$$\min \int_{\Delta} \left| \sum_{i=1}^{n} l_i\ (\beta_k)\ h^{(i)}\ (\tau) \right| d\tau > \varepsilon\ (\varkappa) > 0 \quad \text{for } l \cdot x^\circ = 1$$

uniformly for all Δ from $[0,T]$, satisfying the condition $\mu(\Delta) > \varkappa > 0$. However, if $\mu(\Delta_k) \to 0$, then when (2.14) is fulfilled, inequality (2.17) still cannot be fulfilled since that would signify that

$$\min_l \max_\tau \left\{ \left| \sum_{i=1}^{n} l_i\ (\beta_k)\ h^{(i)}\ (\tau) \right| \Big/ \psi\ (\tau) \right\} = 0 \quad \text{for } l \cdot x^\circ = 1 \qquad (2.18)$$

but under the conditions of complete controllability of system (2.1), (2.18) cannot be satisfied. Consequently, for large values of β the inequality $\alpha(\beta) < \beta$ is satisfied. But this means that there exists a number β° satisfying the conditions (1.12) and (1.13). Consequently, for the problem being considered there exists an optimum control $u^\circ(t)$ which is determined thus:

$$u^\circ(t) = \omega\ (t,\ \beta^\circ)\ \text{sign} \left(\sum_{i=1}^{n} l_i^\circ h^{(i)}\ (t) \right) \quad \text{for } t \text{ in } \Delta^\circ$$

$$u^\circ(t) = 0 \quad \text{for } t \text{ outside } \Delta^\circ \qquad (2.19)$$

Here l_i° and Δ° are solutions of problem (2.16) for the value $\beta = \beta^\circ$ satisfying conditions (1.12) and (1.13).

Problem (2.16) can be solved numerically by descent along the magnitudes $\{l_i\}$ since in a wide class of cases the set Δ in (2.14) has a simple structure and consists of a small number of segments from $[0,T]$.

3. As an illustrative example let us consider the problem of damping the linear oscillator

$$\frac{d^2x}{dt^2} + \varkappa^2 x = u \qquad (\varkappa = \text{const}) \qquad (3.1)$$

within the time T of one period of its natural oscillations, $T = 2\pi/\varkappa$. Let us here require the minimization of the quantity

$$\max \left[\max_\tau u^2\ (\tau),\ \nu \int_0^T |\ u\ (\tau)\ |\ d\tau \right] = \min_u \qquad (\nu > 0 = \text{const}) \qquad (3.2)$$

N o t e 3.1 . As above we consider here, instead of the problem of damping the system (3.2) from the state $x(0) = x^\circ$ to the state $x(T) = 0$, the problem of accelerating the system (3.2) from the equilibrium state $x(0) = 0$ to the state $x(T) = x^\circ$. The optimum control $u^\circ(\vartheta)$ of the original problem is obtained from the solution $u^\circ(t)$ of the auxiliary problem by transforming the interval $0 \leqslant t \leqslant T$ to the interval $0 \leqslant \vartheta \leqslant T$ by substitution $\vartheta = T - t$.

In the form of system (2.1), Equation (3.1) is

$$dx_1 / dt = x_2, \qquad dx_2 / dt = -\varkappa^2 x_1 + u \tag{3.3}$$

The fundamental matrix $F(t)$ of system (3.3) is defined by the equality

$$F(t) = \{f_{ij}(t)\} = \begin{pmatrix} \cos\varkappa t & \varkappa^{-1}\sin\varkappa t \\ -\varkappa\sin\varkappa t & \cos\varkappa t \end{pmatrix} \tag{3.4}$$

In the given case the function $\omega(t,\beta)$ is

$$\omega(t,\beta) = \omega(\beta) = \sqrt{\beta} \tag{3.5}$$

Therefore, in the given case problem (2.16) reduces to the problem

$$\gamma(\beta) = \min_l \max_\Delta \left[\int_\Delta \frac{1}{\sqrt{\beta}}\left| -\frac{l_1}{\varkappa}\sin\varkappa\tau + l_2\cos\varkappa\tau\right| d\tau\right] \qquad \text{for } l_1x_{10} + l_2x_{20} = 1$$

$$\mu(\Delta) = \min\left(\frac{\sqrt{\beta}}{\nu}, \frac{2\pi}{\varkappa}\right) \tag{3.6}$$

The minimum in the left-hand side of (3.6) is reached under the condition

$$\left(\frac{l_1}{\varkappa}\right)^2 + l_2^2 = \min \qquad \text{for} \quad l_1x_{10} + l_2x_{20} = 1$$

i.e. when

$$l_1(\beta) = \frac{\varkappa^2 x_{10}}{\varkappa^2 x_{10}^2 + x_{20}^2} \qquad l_2(\beta) = \frac{x_{20}}{\varkappa^2 x_{10}^2 + x_{20}^2} \tag{3.7}$$

$$\Delta(\beta) = \{\Delta_1(\beta), \Delta_2(\beta)\}, \quad \text{if} \quad \frac{\sqrt{\beta}}{\nu} < \frac{2\pi}{\varkappa} \tag{3.8}$$

$$\Delta(\beta) = \left[0, \frac{2\pi}{\varkappa}\right] \quad \text{if} \quad \frac{\sqrt{\beta}}{\nu} \geqslant \frac{2\pi}{\varkappa} \tag{3.9}$$

Here

$$\Delta_1 = \left[t_* + \frac{\pi}{2\varkappa} - \frac{\sqrt{\beta}}{4\nu},\ t_* + \frac{\pi}{2\varkappa} + \frac{\sqrt{\beta}}{4\nu}\right]$$

$$\Delta_2 = \left[t_* + \frac{\pi 3}{2\varkappa} - \frac{\sqrt{\beta}}{4\nu},\ t_* + \frac{3\pi}{2\varkappa} + \frac{\sqrt{\beta}}{4\nu}\right]$$

$$t_* = -\frac{\xi}{\varkappa}, \quad \cos\xi = -\frac{\varkappa x_{10}}{(\varkappa^2 x_{10}^2 + x_{20}^2)^{1/2}}, \qquad \sin\xi = \frac{x_{20}}{(\varkappa^2 x_{10}^2 + x_{20}^2)^{1/2}}$$

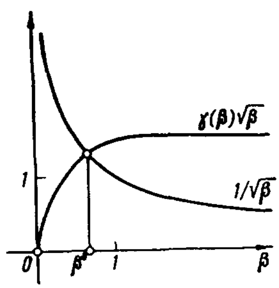

Fig. 1

The minimum $\gamma(\beta)$ is determined by the equalities

$$\gamma(\beta) = 4\int_0^{\sqrt{\beta}/4\nu} \frac{\cos\varkappa\tau\, d\tau}{\sqrt{\beta(\varkappa^2 x_{10}^2 + x_{20}^2)}} = \frac{4\sin[\varkappa\sqrt{\beta}/4\nu]}{\varkappa\sqrt{\beta(\varkappa^2 x_{10}^2 + x_{20}^2)}} \qquad \text{if} \quad \frac{\sqrt{\beta}}{\nu} \leqslant \frac{2\pi}{\varkappa} \tag{3.10}$$

$$\gamma(\beta) = 4\int_0^{\pi/2\varkappa} \frac{\cos\varkappa\tau\, d\tau}{\sqrt{\beta(\varkappa^2 x_{10}^2 + x_{20}^2)}} = \frac{4}{\varkappa\sqrt{\beta(\varkappa^2 x_{10}^2 + x_{20}^2)}} \qquad \text{if} \quad \frac{\sqrt{\beta}}{\nu} \geqslant \frac{2\pi}{\varkappa} \tag{3.11}$$

The number β° satisfying conditions (1.12) and (1.13) is consequently determined as the smallest root of Equation

$$\beta\gamma(\beta) = 1 \tag{3.12}$$

where the function $\gamma(\beta)$ is defined by Equations (3.10) and (3.11).

A graphical solution of Equation (3.12) is shown in Fig.1.

Thus, the optimum control $u^\circ(t)$ has the form

$$u^\circ(t) = \sqrt{\beta^\circ}\,\mathrm{sign}\,[\sin \varkappa (t - t_*)] \quad \text{for} \quad \begin{cases} \left| t - t_* - \frac{\pi}{2\varkappa} \right| < \min\left[\frac{\sqrt{\beta^\circ}}{4\nu}, \frac{2\pi}{\varkappa}\right] \\ \left| t - t_* - \frac{3\pi}{2\varkappa} \right| < \min\left[\frac{\sqrt{\beta^\circ}}{4\nu}, \frac{2\pi}{\varkappa}\right] \end{cases}$$

$$u^\circ(t) = 0 \quad \text{for other} \quad t$$

Here the number t_* is determined from the equality

$$t_* = -\frac{\zeta}{\varkappa}, \quad \cos\zeta = -\frac{\varkappa x_{10}}{(\varkappa^2 x_{10}^2 + x_{20}^2)^{1/2}}, \quad \sin\zeta = \frac{x_{20}}{(\varkappa^2 x_{10}^2 + x_{20}^2)^{1/2}}$$

Note 3.2. In the case (3.8) if the point $\tau < 0$ falls inside the segment $\Delta_1(\beta)$, then the part of this segment corresponding to the values $\tau < 0$ is carried over to the right inside $[0,T]$ by the magnitude T of the period; if however, in the case (3.8) the point $\tau = T$ falls inside $\Delta_2(\beta)$, then the part of this segment corresponding to the values $\tau < T$ is carried over to the left inside $[0,T]$ by the magnitude T of the period.

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